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*Published in:*  
Linear Algebra and its Applications

*Publication date:*  
2007

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*  
van Dam, E. R., Haemers, W. H., & Koolen, J. H. (2007). Cospectral graphs and the generalized adjacency matrix. *Linear Algebra and its Applications*, 423(1), 33-41. <http://dx.doi.org/10.1016/j.laa.2006.07.017>

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# Cospectral graphs and the generalized adjacency matrix

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Received 12 May 2006; accepted 31 July 2006

Available online 11 September 2006

Submitted by D.M. Cvetković

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## Abstract

Let  $J$  be the all-ones matrix, and let  $A$  denote the adjacency matrix of a graph. An old result of Johnson and Newman states that if two graphs are cospectral with respect to  $yJ - A$  for two distinct values of  $y$ , then they are cospectral for all  $y$ . Here we will focus on graphs cospectral with respect to  $yJ - A$  for exactly one value  $\hat{y}$  of  $y$ . We call such graphs  $\hat{y}$ -cospectral. It follows that  $\hat{y}$  is a rational number, and we prove existence of a pair of  $\hat{y}$ -cospectral graphs for every rational  $\hat{y}$ . In addition, we generate by computer all  $\hat{y}$ -cospectral pairs on at most nine vertices. Recently, Chesnokov and the second author constructed pairs of  $\hat{y}$ -cospectral graphs for all rational  $\hat{y} \in (0, 1)$ , where one graph is regular and the other one is not. This phenomenon is only possible for the mentioned values of  $\hat{y}$ , and by computer we find all such pairs of  $\hat{y}$ -cospectral graphs on at most eleven vertices.

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*AMS classification:* 05C50; 05E99

*Keywords:* Cospectral graphs; Generalized spectrum; Generalized adjacency matrix

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<sup>1</sup> The research of E.R. van Dam has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

<sup>2</sup> This work was done while J.H. Koolen was visiting Tilburg University and he appreciates the hospitality and support received from Tilburg University.

## 1. Introduction

For a graph  $\Gamma$  with adjacency matrix  $A$ , any matrix of the form  $M = xI + yJ + zA$  with  $x, y, z \in \mathbb{R}, z \neq 0$  is called a *generalized adjacency matrix* of  $\Gamma$  (as usual,  $J$  is the all-ones matrix and  $I$  the identity matrix). Since we are interested in the relation between  $\Gamma$  and the spectrum of  $M$ , we can restrict to generalized adjacency matrices of the form  $yJ - A$  without loss of generality.

Let  $\Gamma$  and  $\Gamma'$  be graphs with adjacency matrices  $A$  and  $A'$ , respectively. Johnson and Newman [7] proved that if  $yJ - A$  and  $yJ - A'$  are cospectral for two distinct values of  $y$ , then they are cospectral for all  $y$ , and hence they are cospectral with respect to all generalized adjacency matrices. If this is the case we will call  $\Gamma$  and  $\Gamma'$   $\mathbb{R}$ -cospectral. So if  $yJ - A$  and  $yJ - A'$  are cospectral for some but not all values of  $y$ , they are cospectral for exactly one value  $\hat{y}$  of  $y$ . Then we say that  $\Gamma$  and  $\Gamma'$  are  $\hat{y}$ -cospectral. Thus cospectral graphs (in the usual sense) are either 0-cospectral or  $\mathbb{R}$ -cospectral. For both types of cospectral graphs, many examples are known (see for example [5]). In Fig. 1 we give an example of both. This figure also gives examples of  $\hat{y}$ -cospectral graphs for  $\hat{y} = \frac{1}{3}$  and  $\hat{y} = -1$ . Note that  $\Gamma$  and  $\Gamma'$  are  $\hat{y}$ -cospectral if and only if their complements are  $(1 - \hat{y})$ -cospectral. So we also have examples for  $\hat{y} = 1, \frac{2}{3}$  and 2. If  $\hat{y} = \frac{1}{2}$ , one can construct a graph cospectral with a given graph  $\Gamma$  by multiplying some rows and the corresponding columns of  $\frac{1}{2}J - A$  by  $-1$ . The corresponding operation in  $\Gamma$  is called *Seidel switching*. This shows that every graph with at least two vertices has a  $\frac{1}{2}$ -cospectral mate.

It is well known that, with respect to the adjacency matrix, a regular graph cannot be cospectral with a nonregular one (see [2, p. 94]). In [5] this result is extended to generalized adjacency matrices  $yJ - A$  with  $y \notin (0, 1)$ . In [1] a regular-nonregular pair of  $\hat{y}$ -cospectral graphs is constructed for all rational  $\hat{y} \in (0, 1)$ . In the next section we shall see that  $\hat{y}$  is rational for any pair of  $\hat{y}$ -cospectral graphs. Thus we have:

**Theorem 1.** *There exists a pair of  $\hat{y}$ -cospectral graphs, where one graph is regular and the other one is not, if and only if  $\hat{y}$  is a rational number satisfying  $0 < \hat{y} < 1$ .*

In the final section we will generate all regular-nonregular  $\hat{y}$ -cospectral pairs on at most eleven vertices. The smallest such pair has only six vertices; it is the  $\frac{1}{3}$ -cospectral pair of Fig. 1. In Section 3 we shall construct  $\hat{y}$ -cospectral graphs for every rational value of  $\hat{y}$ . Therefore:

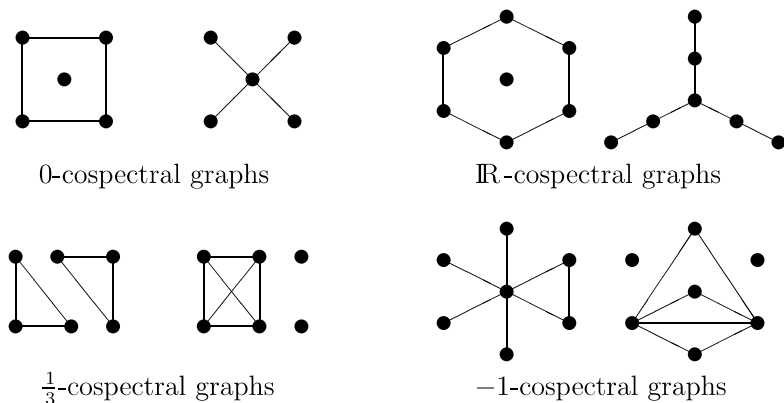


Fig. 1. Some examples of (generalized) cospectral graphs.

**Theorem 2.** *There exists a pair of  $\hat{y}$ -cospectral graphs if and only if  $\hat{y}$  is a rational number.*

In the final section we also generate all pairs of  $\hat{y}$ -cospectral graphs on at most nine vertices.

## 2. The generalized characteristic polynomial

For a graph  $\Gamma$  with adjacency matrix  $A$ , the polynomial  $p(x, y) = \det(xI + yJ - A)$  will be called the *generalized characteristic polynomial* of  $\Gamma$ . Thus  $p(x, y)$  has integral coefficients,  $p(x, y)$  can be interpreted as the characteristic polynomial of  $A - yJ$ , and  $p(x, 0) = p(x)$  is the characteristic polynomial of  $A$ . The generalized characteristic polynomial is closely related to the so-called *idiosyncratic polynomial*, which was introduced by Tutte [9] as the characteristic polynomial of  $A + y(J - I - A)$ . We prefer the polynomial  $p(x, y)$ , because it has the important property that the degree in  $y$  is only 1. Indeed, for an arbitrary square matrix  $M$  it is known that  $\det(M + yJ) = \det M + y\Sigma \operatorname{adj} M$ , where  $\Sigma \operatorname{adj} M$  denotes the sum of the entries of the adjugate (adjoint) of  $M$ . It is also easily derived from the fact that by Gaussian elimination in  $xI + yJ - A$  one can eliminate all  $y$ -s, except for those in the first row. In this way we will obtain more useful expressions for  $p(x, y)$  as follows. Partition  $A$  according to a vertex  $v$ , the neighbors of  $v$  and the remaining vertices ( $\mathbf{1}$  denotes an all-ones vector, and  $\mathbf{0}$  an all-zeros vector):

$$A = \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{0}^\top \\ \mathbf{1} & A_1 & B \\ \mathbf{0} & B^\top & A_0 \end{bmatrix}.$$

Then

$$\begin{aligned} p(x, y) &= \det \begin{bmatrix} x + y & (y - 1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (y - 1)\mathbf{1} & xI + yJ - A_1 & yJ - B \\ y\mathbf{1} & yJ - B^\top & xI + yJ - A_0 \end{bmatrix} \\ &= \det \begin{bmatrix} x + y & (y - 1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (-1 - x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} \\ &= p(x) + y \det \begin{bmatrix} 1 & \mathbf{1}^\top & \mathbf{1}^\top \\ (-1 - x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} \\ &= p(x) + y \det \begin{bmatrix} 1 & 2\mathbf{1}^\top & \mathbf{0}^\top \\ -\mathbf{1} & xI - A_1 & J - B \\ \mathbf{0} & J - B^\top & xI - A_0 \end{bmatrix} - xy \det \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & xI - A_1 & -B \\ \mathbf{1} & -B^\top & xI - A_0 \end{bmatrix} \end{aligned}$$

This expression provides the coefficients of the three highest powers of  $x$  in  $p(x, y)$ . A similar expression is used for the computations in Section 4.

**Lemma 1.** *Let  $\Gamma$  be a graph with  $n$  vertices,  $e$  edges, and generalized characteristic polynomial  $p(x, y) = \sum_{i=0}^n (a_i + b_i y)x^i$ . Then  $a_n = 1$ ,  $b_n = 0$ ,  $a_{n-1} = 0$ ,  $b_{n-1} = n$ ,  $a_{n-2} = -e$  and  $b_{n-2} = 2e$ .*

**Proof.** By using the above expression for  $p(x, y)$ , and straightforward calculations.  $\square$

Thus the coefficient of  $x^{n-2}$  in  $p(x, y)$  equals  $e(2y - 1)$ . This implies the known fact that, for any  $y \neq \frac{1}{2}$ , the number of edges of a graph can be deduced from the spectrum of  $yJ - A$ . Note that a  $\frac{1}{2}$ -cospectral pair with distinct numbers of edges can easily be made by Seidel switching.

Now let  $\Gamma$  and  $\Gamma'$  be graphs with generalized characteristic polynomials  $p(x, y)$  and  $p'(x, y)$ , respectively. It is clear that  $p(x, y) \equiv p'(x, y)$  if and only if  $\Gamma$  and  $\Gamma'$  are  $\mathbb{R}$ -cospectral, and  $\Gamma$  and  $\Gamma'$  are  $\hat{y}$ -cospectral if and only if  $p(x, \hat{y}) = p'(x, \hat{y})$  for all  $x \in \mathbb{R}$ , whilst  $p(x, y) \not\equiv p'(x, y)$ . If this is the case, then  $a_i + \hat{y}b_i = a'_i + \hat{y}b'_i$  with  $(a_i, b_i) \neq (a'_i, b'_i)$  for some  $i$  ( $0 \leq i \leq n - 3$ ). This implies  $\hat{y} = -(a_i - a'_i) / (b_i - b'_i)$ . Thus we proved the mentioned result of Johnson and Newman, that there is only one possible value of  $\hat{y}$ . Moreover, we see:

**Proposition 1.** *Let  $\Gamma$  and  $\Gamma'$  be two  $\hat{y}$ -cospectral graphs then*

(i)  $\hat{y}$  is a rational number.

(ii) Let  $|\hat{y}| = p/q$  with  $p$  and  $q$  relative primes. Then  $|\hat{y}| \leq p \leq 4 \left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$ .

**Proof.** We have  $\hat{y} = -(a_i - a'_i) / (b_i - b'_i)$  for some  $i$  with  $a_i, a'_i, b_i$  and  $b'_i$  integral. Therefore  $p \leq |a_i| + |a'_i|$ . The Hadamard bound gives that the absolute value of the determinant of any  $m \times m$   $(0, 1)$ -matrix is at most  $2^{-m}(m+1)^{(m+1)/2}$ . Hence the coefficient  $a_i$  of the characteristic polynomial  $\sum_{i=0}^n a_i x^i$  of any  $n \times n$   $(0, 1)$ -matrix satisfies

$$|a_i| \leq \binom{n}{i} 2^{i-n}(n-i+1)^{\frac{n-i+1}{2}} \leq \sum_{i=0}^n \binom{n}{i} 2^{i-n}(n+1)^{\frac{n-i+1}{2}} = 2^{-n} (2 + \sqrt{n+1})^{n+1}.$$

Therefore  $p \leq 4 \left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$ .  $\square$

The generalized characteristic polynomial  $p(x, y)$  of a graph  $\Gamma$  is related to the set of main angles  $\{\beta_1, \dots, \beta_\ell\}$  of  $\Gamma$ . Suppose the adjacency matrix  $A$  of  $\Gamma$  has  $\ell$  distinct eigenvalues  $\lambda_1 > \dots > \lambda_\ell$  with multiplicities  $m_1, \dots, m_\ell$ , respectively, then the main angle  $\beta_i$  is defined as the cosine of the angle between the all-ones vector  $\mathbf{1}$  and the eigenspace of  $\lambda_i$ . For  $i = 1, \dots, \ell$ , let  $V_i$  be an  $n \times m_i$  matrix whose columns are an orthonormal basis for the eigenspace of  $\lambda_i$ . Then  $\beta_i \sqrt{n} = \|V_i^\top \mathbf{1}\|$ . Moreover, we can choose  $V_i$  such that  $V_i^\top \mathbf{1} = \beta_i \sqrt{n} \mathbf{e}_1$  (where  $\mathbf{e}_1$  is the unit vector in  $\mathbb{R}^{m_i}$ ). Put  $V = [V_1 \dots V_\ell]$ , then  $V^\top AV = A$ , where  $A$  is the diagonal matrix with the spectrum of  $A$ , and  $V^\top \mathbf{1} = \sqrt{n} [\beta_1 \mathbf{e}_1^\top \dots \beta_\ell \mathbf{e}_\ell^\top]^\top$ .

Assume that  $\Gamma$  and  $\Gamma'$  are cospectral graphs with the same angles. Then there exist matrices  $V$  and  $V'$  such that  $V^\top AV = V'^\top A' V' = A$  and  $V^\top \mathbf{1} = V'^\top \mathbf{1}$ . Define  $Q = VV'^\top$ , then  $Q^\top A Q = A'$  and  $Q\mathbf{1} = Q^\top \mathbf{1} = \mathbf{1}$ . This implies that  $Q^\top (yJ - A)Q = yJ - A'$ , so  $yJ - A$  and  $yJ - A'$  are cospectral for every  $y \in \mathbb{R}$ , hence  $\Gamma$  and  $\Gamma'$  have the same generalized characteristic polynomial.

Cvetković and Rowlinson [3] (see also [4, p. 100]) proved the following expression for  $p(x, y)$  in terms of the spectrum and the main angles of  $\Gamma$ :

$$p(x, y) = p(x) \left( 1 + yn \sum_{i=1}^{\ell} \beta_i^2 / (x - \lambda_i) \right).$$

This formula also shows that the main angles can be obtained from  $p(x, y)$ , as can be seen as follows. Suppose  $q(x) = \prod_{i=1}^{\ell} (x - \lambda_i)$  is the minimal polynomial of  $A$ , put  $r(x) = p(x)/q(x)$  and

$q(x, y) = p(x, y)/r(x)$ . Then  $q(x, y)$  is a polynomial satisfying  $q(\lambda_j, 1) = n\beta_j^2 \prod_{i \neq j} (\lambda_j - \lambda_i)$ , which proves the claim. As a consequence, we also proved (a result due to Johnson and Newman [7]) that  $\Gamma$  and  $\Gamma'$  are  $\mathbb{R}$ -cospectral if and only if there exist an orthogonal matrix  $Q$  such that  $Q^\top A Q = A'$  and  $Q\mathbf{1} = \mathbf{1}$ . The next theorem recapitulates the conditions we have seen for graphs being  $\mathbb{R}$ -cospectral.

**Theorem 3.** *Let  $\Gamma$  and  $\Gamma'$  be graphs with adjacency matrices  $A$  and  $A'$ . Then the following are equivalent:*

- $\Gamma$  and  $\Gamma'$  have identical generalized characteristic polynomials,
- $\Gamma$  and  $\Gamma'$  are cospectral with respect to all generalized adjacency matrices,
- $\Gamma$  and  $\Gamma'$  are cospectral, and so are their complements,
- $\Gamma$  and  $\Gamma'$  are cospectral, and have the same main angles,
- $yJ - A$  and  $yJ - A'$  are cospectral for two distinct values of  $y$ ,
- $yJ - A$  and  $yJ - A'$  are cospectral for any irrational value of  $y$ ,
- $yJ - A$  and  $yJ - A'$  are cospectral for any  $y$  with  $|y| > 4 \left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$ ,
- there exist an orthogonal matrix  $Q$ , such that  $Q^\top A Q = A'$  and  $Q\mathbf{1} = \mathbf{1}$ .

### 3. A construction

We construct pairs of graphs  $\Gamma$  and  $\Gamma'$  on  $n$  vertices. For each pair the vertex set is partitioned into three parts with sizes  $a, b$ , and  $c$  for  $\Gamma$ , and  $a', b'$ , and  $c' = c$  for  $\Gamma'$ . Thus  $a + b = a' + b' = n - c$ . With these partitions  $\Gamma$  and  $\Gamma'$  are defined via their adjacency matrices  $A$  and  $A'$  as follows ( $O$  denotes the all-zeros matrix):

$$A = \begin{bmatrix} O & O & O \\ O & O & J \\ O & J & J - I \end{bmatrix}, \quad A' = \begin{bmatrix} O & J & O \\ J & O & J \\ O & J & J - I \end{bmatrix}.$$

So for the matrices  $M = yJ - A$  and  $M' = yJ - A'$  we get:

$$M = \begin{bmatrix} yJ & yJ & yJ \\ yJ & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}, \quad M' = \begin{bmatrix} yJ & (y-1)J & yJ \\ (y-1)J & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}.$$

Clearly  $\text{rank}(M) \leq c + 2$ , so the characteristic polynomial  $p(x, y)$  of  $M$  has a factor  $x^{n-c-2}$ . Moreover  $\text{rank}(M - I) \leq a + b + 1$ , so  $p(x, y)$  has a factor  $(x - 1)^{c-1}$ . In a similar way we find that the characteristic polynomial  $p'(x, y)$  of  $M'$  also has a factor  $x^{n-c-2}(x - 1)^{c-1}$ . Define

$$r(x, y) = \frac{p(x, y)}{x^{n-c-2}(x - 1)^{c-1}} \quad \text{and} \quad r'(x, y) = \frac{p'(x, y)}{x^{n-c-2}(x - 1)^{c-1}}.$$

Then  $r(x, y)$  and  $r'(x, y)$  are polynomials of degree 3 in  $x$  and degree 1 in  $y$ . Clearly  $M$  and  $M'$  are cospectral if  $r(x, y) = r'(x, y)$  for all  $x \in \mathbb{R}$ . Write

$$r(x, y) = t_0 + t_1x + t_2x^2 + t_3x^3, \quad \text{and} \quad r'(x, y) = t'_0 + t'_1x + t'_2x^2 + t'_3x^3,$$

where  $t_i$  and  $t'_i$  are linear functions in  $y$ . Then  $t_3 = t'_3 = 1$ , and  $t_2 = t'_2 = -ny + c - 1$ , because  $-ny = -\text{trace}(M) = -\text{trace}(M')$ , which equals the coefficient of  $x^{n-1}$  in  $p(x, y)$  and  $p'(x, y)$ . We shall require that

$$b'(a' + c) = bc.$$

This means that  $\Gamma$  and  $\Gamma'$  have the same number of edges. In the previous section we saw that the number of edges determines the coefficient of  $x^{n-2}$  in the generalized characteristic polynomial. Therefore the above requirement gives  $t_1 = t'_1$ . Finally we shall use the fact that  $r(x, y)$  and  $r'(x, y)$  are the characteristic polynomials of the quotient matrices  $R$  and  $R'$  of  $M$  and  $M'$ , respectively (the quotient matrices are the  $3 \times 3$  matrices consisting of the row sums of the blocks). So if we choose  $y = \hat{y}$  such that these quotient matrices have the same determinant we have  $t_0 = t'_0$ , and therefore  $M$  and  $M'$  have the same spectrum. We find

$$\det R = \det \begin{bmatrix} ya & yb & yc \\ ya & yb & yc - c \\ ya & yb - b & yc - c + 1 \end{bmatrix} = -yabc,$$

and

$$\det R' = \det \begin{bmatrix} ya' & yb' - b' & yc \\ ya' - a' & yb' & yc - c \\ ya' & yb' - b' & yc - c + 1 \end{bmatrix} = (1 - 2y)a'b'(c - 1).$$

Using  $bc = b'(a' + c)$ , this leads to  $\hat{y} = a'(c - 1)/(2a'c - 2a' - ac - aa')$ . So any choice of positive integers  $a, a', b, b'$ , and  $c$  that satisfy  $a + b = a' + b'$ ,  $bc = b'(a' + c)$ , and  $2a'c - 2a' - ac - aa' \neq 0$  leads to a pair of  $\hat{y}$ -cospectral graphs with the above  $\hat{y}$  (indeed,  $\hat{y}$  is uniquely determined, hence  $\Gamma$  and  $\Gamma'$  are not  $\mathbb{R}$ -cospectral). For example  $(a, a', b, b', c) = (2, 4, 3, 1, 2)$  leads to the two  $-1$ -cospectral graph of Fig. 1. Moreover, by a suitable choice of these numbers we can get every rational value of  $\hat{y} > 1/2$ . Indeed, write  $\hat{y} = p/q$  with  $2p - q \geq 2$ , then

$$a = 2p - q - 1, \quad a' = a(p + 1), \quad b = p(a + 1), \quad b' = p, \quad \text{and} \quad c = p + 1$$

satisfy the required conditions and gives  $\hat{y} = p/q$ . As remarked in the introduction,  $\frac{1}{2}$ -cospectral graphs are easily made by use of Seidel switching, and we also saw that two graphs are  $\hat{y}$ -cospectral if and only if their complements are  $(1 - \hat{y})$ -cospectral. Thus we have:

**Proposition 2.** *A pair of  $\hat{y}$ -cospectral graphs exists for every rational  $\hat{y}$ .*

Variations on the above construction are possible. The  $\hat{y}$ -cospectral pairs, with  $0 < \hat{y} < 1$ , constructed in [1] (where one graph is regular and the other one not) are of a completely different nature.

#### 4. Computer enumeration

By computer we enumerated all graphs with a  $\hat{y}$ -cospectral ( $\hat{y} \neq \frac{1}{2}$ ) mate on at most nine vertices. For fixed numbers of vertices ( $n$ ) and edges ( $e$ ) we generated all graphs with these numbers using *nauty* [8], and for each graph we computed  $p(x, y)$  for  $x = 0, \dots, n$ . Note that these  $n + 1$  linear functions in  $y$  uniquely determine the polynomial  $p(x, y)$ . For each pair of graphs we compared the corresponding linear functions, giving a system of  $n + 1$  linear equations in  $y$ . If the system had infinitely many solutions, then we concluded that the pair was  $\mathbb{R}$ -cospectral; and if it had a unique solution  $\hat{y}$ , then the pair was  $\hat{y}$ -cospectral. The results of these computations are given in Table 1. Note that we only considered the cases where  $2e \leq \binom{n}{2}$  since, as mentioned before, the complement of a pair of  $\hat{y}$ -cospectral graphs is a pair of  $(1 - \hat{y})$ -cospectral graphs. In the table, the columns with  $e$  give the numbers of edges and the columns with  $\#$  give the numbers of graphs

Table 1  
Numbers of graphs with a  $\hat{y}$ -cospectral mate

Graphs on 5 vertices			Graphs on 6 vertices			Graphs on 7 vertices			Graphs on 8 vertices		
$e$	#	$\mathbb{R} 0$	$e$	#	$\mathbb{R} 0 \frac{1}{3}$	$e$	#	$\mathbb{R} -1$	$e$	#	$\mathbb{R} -1$
2	2	..	2	2	..	2	2	.	2	2	.
3	4	..	3	5	..	3	5	.	3	5	.
4	6	.2	4	9	.4	4	10	.	4	11	.
5	6	..	5	15	.2	5	21	.	5	24	.
			6	21	.2	6	41	2	6	56	2
			7	24	.2	7	65	2	7	115	4
						8	97	4	8	221	10
						9	131	6	9	402	30
						10	148	6	10	663	55

Graphs on 9 vertices		
$e$	#	$\mathbb{R} -2$
2	2	..
3	5	..
4	11	..
5	25	..
6	63	..
7	148	..
8	345	..
9	771	..
10	1637	..
11	3252	..
12	5995	..
13	10120	..
14	15615	..
15	21933	..
16	27987	..
17	32403	..
18	34040	..



Table 2  
Numbers of regular graphs  $\hat{y}$ -cospectral with nonregular graphs

$n$	$k$	#	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$
6	2	2	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
8	2	3	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
8	3	6	1	.	.	3	.	.	.	.	.	.	.	.	.	.	.
9	2	4	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.
9	4	16	.	1	.	1	.	.	2	1	2	.	2	1	2	1	1
10	2	5	2	1	.	1	.	.	.	.	.	.	.	.	.	.	.
10	3	21	1	2	1	5	.	.	1	.	1	.	.	.	.	.	.
10	4	60	.	4	.	3	.	.	4	.	.	1	1	.	1	1	.
11	4	266	.	45	.	22	1	2	5	.	.	.	.	.	.	1	.

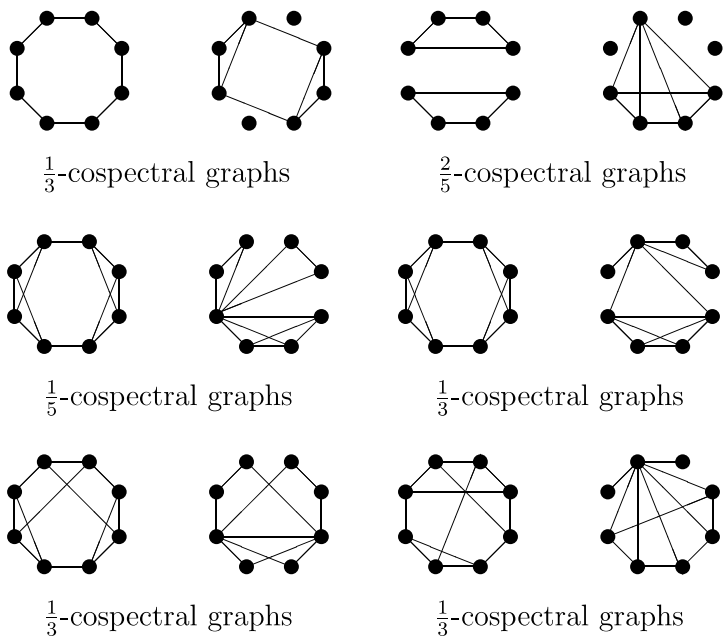


Fig. 2. Regular-nonregular  $\hat{y}$ -cospectral pairs on eight vertices.

with  $e$  edges. The columns with header  $\mathbb{R}$  contain the number of graphs that have an  $\mathbb{R}$ -cospectral mate. The columns with a number  $\hat{y}$  in the header contain the numbers of graphs that have a  $\hat{y}$ -cospectral mate. Note that this does not mean that a graph cannot be counted in more than one column; for example, of the triple of 0-cospectral graphs with seven vertices and five edges, one (the union of  $K_{1,4}$  and an edge) also has a 1-cospectral mate, and another (the union of  $K_{2,2}$ , an isolated vertex, and an edge) also has a  $\frac{1}{4}$ -cospectral mate.

We may conclude that the 0-cospectral pair of Fig. 1 is the smallest pair of  $\hat{y}$ -cospectral graphs. The smallest pair of  $\hat{y}$ -cospectral graphs for  $\hat{y} \neq 0$  is the pair of  $\frac{1}{3}$ -cospectral graphs in Fig. 1. This is also the smallest example where one graph is regular, and the other one not. The smallest  $\mathbb{R}$ -cospectral pair of graphs, and the smallest  $\hat{y}$ -cospectral pair of graphs for a negative  $\hat{y}$  are also given in Fig. 1.

We remark that also in [6] all graphs with an  $\mathbb{R}$ -cospectral mate, as well as all graphs with a (usual) cospectral mate were enumerated (up to eleven vertices). The latter enumeration is different from our enumeration of graphs with a 0-cospectral mate since it also counts graphs with a  $\mathbb{R}$ -cospectral mate (whereas these are excluded in our enumeration).

We also enumerated all regular graphs with a nonregular  $\hat{y}$ -cospectral mate ( $\hat{y} \neq \frac{1}{2}$ ) on at most eleven vertices; see Table 2. The columns with a number  $\hat{y}$  in the header contain the numbers of graphs that have a  $\hat{y}$ -cospectral mate. The column with  $n$  gives the number of vertices, the column with  $k$  gives the valency and the column with # gives the number of  $k$ -regular graphs with  $n$  vertices. The computations were restricted to  $v \geq 2k + 1$  for a similar reason as before. We remark further that for missing pairs  $(v, k)$  in the considered range, such as  $(11, 2)$ , there are no regular graphs with a  $\hat{y}$ -cospectral mate ( $\hat{y} \neq \frac{1}{2}$ ). In Fig. 2 we give all regular-nonregular  $\hat{y}$ -cospectral pairs on eight vertices (up to complements).

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